# Multiattribute Nonlinear Pricing of Network Goods

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**Summary**

We consider a general multiattribute screening model with demand externalities where the principal can design contract instruments (e.g., quality and capacity) of arbitrary dimension such as to independently influence the gross consumption value of each product and the probability with which it is actually consumed. Both the gross consumption value and the respective consumption probability for each product are subject to network externalities in the sense that they depend on the anticipated number of consumers who self-select to purchase the same product, a different product, or no product. We characterize the class of all implementable menus and provide a full set of necessary optimality conditions for the underlying general multiattribute fulfilled-expectations screening problem. We also give sufficient conditions, some of which generalize the Spence-Mirrlees single-crossing requirement, that guarantee a simple pointwise optimal solution. In contrast to standard single-attribute nonlinear pricing, an optimal multiattribute menu can be nonmonotonic in the consumers' type. Our framework applies to situations where a monopolist seeks a profit-maximizing vertically differentiated product portfolio and accompanying optimal infrastructure investments in the presence of demand externalities. Concrete examples include networked on-demand systems, the design of software product lines, as well as insurance contracts with mitigation clauses.

**Keywords:** Hidden-Information; Network Externalities; Screening.

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# 1 Introduction

Purchase decisions are rarely made in isolation. When confronted with a menu of differentiated substitute products, a consumer often forms beliefs about who else is likely to purchase the same, another or no product at all, which may significantly affect her expected consumption utility and thus her choice. Such demand externalities generally exist for goods whose usage value depends on overall adoption, such as communication devices (e.g., fax, phone), computer operating systems, software programs, fashion goods, insurance against natural disasters, or tickets to funfair rides. We refer to *any* product whose value depends at least for some consumers on the purchase behavior of others as a *network good*. In fact, it is not easy to think of examples of goods or services whose utility can be determined in a complete vacuum. The value of products depends on the adoption decisions of others because of the network externalities that result when products can be used in conjunction with others, are positional, or when consumers expect to trade them later in a second-hand market, to name only a few possibilities. In addition to these "direct" externalities which influence the fulfilled-expectations utility of the product when it is being used, *actual* consumption might be rationed as a consequence of "indirect" externalities, for instance if availability of the product depends on a common limited resource such as the capacity of a network, the staffing of a call center, or the throughput of a funfair ride. In equilibrium, the overall "fulfilled expectations" utility of a product thus generally depends on both direct and indirect externalities.

A firm's profit-maximizing design of its product line (or "menu") and associated nonlinear pricing needs to take into account interdependent consumer choice behavior due to direct and indirect externalities. We analyze the underlying multiattribute screening problem in a general setting for arbitrary one-dimensional distributions of consumer types.[1] The firm can thereby not only choose "quality" attributes to create a vertically differentiated product line, but it can also select "capacity" attributes affecting the availability of each individual product.

We provide a general framework for the design of an optimal (differentiable) menu of network goods when the firm can use a vector of *quality* attributes $x = (x_1, \ldots, x_n)$, a vector of *capacity* attributes $y = (y_1, \ldots, y_m)$, and a *transfer* $t$ (i.e., the product price) as instruments. We show that, in contrast to the standard single-attribute screening problem, the optimal menu can be nonmonotonic in the consumers' type. Even under the standard single-crossing assumptions it is generally not possible (nor desirable!) to guarantee that higher types choose products of uniformly higher quality and/or capacity. We completely characterize the class of implementable menus and provide necessary optimality conditions which can be used to find an optimal solution to the firm's multiattribute screening problem. We also give sufficient conditions, in part generalizing the Spence-Mirrlees single-crossing requirement, that guarantee a simple pointwise optimal solution to the problem.

# 2 Model Overview and Main Results

Consumers are heterogeneous of types $\theta \in \Theta = [\underline{\theta}, \bar{\theta}]$ (where $\underline{\theta} < \bar{\theta}$), distributed according to the cumulative distribution function $F$. When purchasing a product of attributes $(\hat{x}, \hat{y}, \hat{t}) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}$, a consumer of type $\theta \in \Theta$ obtains the utility

$$u(\hat{x}, \hat{y}, \hat{t}, X(\theta), Y(\theta), \theta) = \left(g(\hat{x}, X(\theta), \theta) - \hat{t}\right) h(\hat{y}, Y(\theta), \theta),$$

[1]Naturally, the type distribution can be multidimensional, if a one-dimensional sufficient statistic for the problem can be found.

where $g : \mathcal{X} \times \mathbb{R} \times \Theta \to \mathbb{R}_+$ and $h : \mathcal{Y} \times \mathbb{R} \times \Theta \to \mathbb{R}_{++}$ are twice differentiable functions and $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{Y} \subset \mathbb{R}^m$ are nonempty convex compact sets. Note that (as we show below) a particular product $(\hat{x}, \hat{y}, \hat{t})$ that a consumer of type $\theta$ selects corresponds to one item of the *menu*

$$(x, y, t) : \Theta \to \mathcal{X} \times \mathcal{Y} \times \mathbb{R},$$

which we assume to be twice differentiable, i.e., $(\hat{x}, \hat{y}, \hat{t}) = (x(\theta), y(\theta), t(\theta))$. The functionals $X, Y : C^2(\Theta, \mathcal{X} \times \mathcal{Y}) \times \Theta \to \mathbb{R}$ map a menu of attributes $(x, y)$ into real numbers that affect a type-$\theta$ consumer's utility. The dependence of $u$ on $X$ and $Y$ expresses how each consumer cares about the consumption choice of others, thus incorporating direct and indirect externalities.

Example. Sundararajan (2003) examines the case when $n = 1$ and $m = 0$ with $h = 1$ where consumers care about total consumption, $X$. In this situation $x(\theta)$ is real-valued and can be interpreted as the quantity purchased by a type-$\theta$ consumer, whence we can write

$$X(x) = \int_\Theta x(\vartheta) dF(\vartheta),$$

independent of $\theta$. More generally, consumers might just care about the amount purchased by inferior types ("downward compatibility"), so that

$$X_d(x, \theta) = \int_{\underline{\theta}}^{\theta} x(\vartheta) dF(\vartheta),$$

or they might only be interested in the amount purchased by higher types ("upward compatibility"), in which case $X_u(x, \theta) = X(x) - X_d(x, \theta)$ becomes the relevant functional. Note that both $X_d$ and $X_u$ depend on $\theta$. □

Our model can accommodate complex externalities where consumers care about any combination of quality and capacity attributes, as long as they can be expressed in terms of a real-valued functional, differentiable in $\theta$.[2] In a *fulfilled-expectations screening equilibrium* where the firm chooses a profit maximizing menu $(x^*, y^*, t^*)$, the equilibrium attribute schedule $(x^*, y^*)$ is correctly anticipated by the consumers, so that then $X^*(\theta) = X(x^*, \theta)$ and $Y^*(\theta) = Y(y^*, \theta)$. In what follows we simplify notation by omitting to explicitly mention the dependence of $X, Y$ on $x, y$ respectively. As is customary in the literature, we consider the firm's multiattribute screening problem in a more abstract mechanism design setting.

Definition 1 *Given $\{\Theta, F, u\}$ a communication mechanism $\Gamma = \langle \mathcal{M}, z \rangle$ consists of a (measurable) message space $\mathcal{M}$ and a mapping $z : \mathcal{M} \to \mathcal{X} \times \mathcal{Y} \times \mathbb{R}$ which assigns an allocation $z(m) = (x, y, t)(m)$ to any message $m \in \mathcal{M}$.*

The firm's preferences over menus $(x, y, t)$ are determined by its *expected profits*,

$$\pi(x, y, t) = \int_\Theta [(t(\vartheta) - c(x(\vartheta)))h(y(\vartheta), Y(\vartheta), \vartheta) - k(\vartheta)]\, dF(\vartheta),$$

where $c : \mathcal{X} \to \mathbb{R}_+$ and $k : \mathcal{Y} \to \mathbb{R}_+$ are increasing, twice differentiable, convex cost functions. The firm's problem of finding a profit-maximizing menu $(x^*, y^*, t^*)$ amounts to determining a communication mechanism that maximizes its expected profits. The following well-known result can be used to substantially simplify the search for an optimal mechanism.

[2]Nothing changes in our results if the externality functionals $X$ and/or $Y$ are vector-valued (at certain points in the paper one would simply have to replace ordinary products of real numbers with the corresponding scalar products).

Proposition 1 (Revelation Principle; Meyerson, 1979) *If for a given mechanism $\Gamma = \langle \mathcal{M}, z \rangle$ a consumer of type $\theta \in \Theta$ finds it optimal to send a message $m^*(\theta)$, then there exists a direct revelation mechanism $\Gamma^d = \langle \Theta, z^d \rangle$, such that $z^d(\theta) = z(m^*(\theta))$ and the consumer finds it optimal to report her type truthfully under $\Gamma^d$.*

Following Guesnerie and Laffont (1984), we assume that consumers decide to participate in the mechanism before their type is known. In other words, participation in the mechanism $\langle \Theta, z \rangle$ is *individually rational* if and only if

$$\int_\Theta u(x(\theta), y(\theta), t(\theta), X(\theta), Y(\theta), \theta) dF(\theta) \geq u_0, \tag{1}$$

for some constant $u_0 \geq 0$ (value of the consumers' outside option). As a consequence of the revelation principle the firm can limit itself to truth-telling (i.e., direct) mechanisms. The following condition of implementability is crucial.

Definition 2 (Implementability) *A direct mechanism $\langle \Theta, z \rangle$ is* implementable *if the direct allocation $z : \Theta \to \mathcal{X} \times \mathcal{Y} \times \mathcal{T}$ satisfies the agent's incentive compatibility (or truth-telling) constraint, i.e., if*

$$u(x(\theta), y(\theta), t(\theta), X(\theta), Y(\theta), \theta) \geq u(x(\hat{\theta}), y(\hat{\theta}), t(\hat{\theta}), X(\theta), Y(\theta), \theta), \tag{2}$$

*for all $\theta, \hat{\theta} \in \Theta$.*

It is now possible to completely characterize all implementable menus (i.e., communication mechanisms) for our general setting.

Proposition 2 (Implementability Criterion) *The twice differentiable menu $(x, y, t) : \Theta \to \mathcal{X} \times \mathcal{Y} \times \mathcal{T}$ is implementable if and only if*

$$\langle g_x - c_x, x' \rangle + \frac{g - c}{h} \langle h_y, y' \rangle = t' \tag{3}$$

*and*

$$\left[ X' \frac{\partial}{\partial X} + Y' \frac{\partial}{\partial Y} + \frac{\partial}{\partial \theta} \right] \left( \langle g_x - c_x, x' \rangle + \frac{g - c}{h} \langle h_y, y' \rangle \right) \geq 0. \tag{4}$$

The implementability criterion is essential in formulating the firm's optimization problem corresponding to its search of a profit-maximizing mechanism. We can use relations (1) and (3) to reformulate the firms' expected profits and obtain the following representation of its multiattribute screening problem.

Proposition 3 (Representation of the Multiattribute Screening Problem) *The firm's profit maximization problem is equivalent to finding a twice differentiable schedule $(x, y) : \Theta \to \mathcal{X} \times \mathcal{Y}$ that solves*

$$\max_{x(\cdot), y(\cdot)} \int_\Theta \left[ (g(x(\theta), X(\theta), \theta) - c(x(\theta)))h(y(\theta), Y(\theta), \theta) - k(y(\theta)) \right] dF(\theta), \tag{5}$$

*subject to (4).*

Using Pontryagin et al.'s (1962) maximum principle (with mixed state-control constraints, cf. Sethi and Thompson (2000, p. 61)) we can provide a full set of necessary optimality conditions which can be used to find solutions of the firm's multiattribute screening problem.

Proposition 4 (Constrained Solution) *Any constrained solution* $(x^*, y^*)$ *to the firm's screening problem (4)–(5) satisfies (with the Lagrangean* $L = (g-c)h - k + \langle \lambda, \xi \rangle + \langle \mu, \upsilon \rangle + \nu \Phi$*)*

$$\lambda' = -L_x(x^*, y^*, X^*, Y^*, \theta, \xi^*, \upsilon^*, \lambda, \mu, \nu), \tag{6}$$

$$\mu' = -L_y(x^*, y^*, X^*, Y^*, \theta, \xi^*, \upsilon^*, \lambda, \mu, \nu), \tag{7}$$

$$(x^{*\prime}, y^{*\prime})(\theta) = (\xi^*, \upsilon^*)(\theta), \tag{8}$$

$$(\xi^*, \upsilon^*)(\theta) \in \arg \max_{(\xi,\upsilon)\in\Xi(\theta)} \left\{ \langle \lambda(\theta), \xi \rangle + \langle \mu(\theta), \upsilon \rangle \right\}, \tag{9}$$

$$\lambda(\underline{\theta}) = \lambda(\bar{\theta}) = 0, \tag{10}$$

$$\mu(\underline{\theta}) = \mu(\bar{\theta}) = 0, \tag{11}$$

*where*

$$\Xi(\theta) = \left\{ (\xi, \upsilon) \in \mathcal{X} \times \mathcal{Y} : \Phi(x^*, y^*, X^*, Y^*, \theta, \xi, \upsilon) \geq 0 \right\},$$

*and*

$$\Phi(x, y, X, Y, \theta, \xi, \upsilon) = \left[ X' \frac{\partial}{\partial X} + Y' \frac{\partial}{\partial Y} + \frac{\partial}{\partial \theta} \right] \left( \langle g_x - c_x, \xi \rangle + \frac{g-c}{h} \langle h_y, \upsilon \rangle \right).$$

The main difficulty in solving the above system of ordinary differential equations is that the constraint set $\Xi$ generally depends on $\theta$ and imposes constraints on both the state variables $(x, y)$ and the control variables $(\xi, \upsilon)$ of this "dynamic system" (interpreting the scalar parameter $\theta$ as "time"). However, ignoring the constraint (4), it may be possible to obtain a pointwise solution to an unconstrained problem.

Proposition 5 (Unconstrained Solution) *Any unconstrained solution* $(x^*, y^*)$ *to the firm's screening problem (5) satisfies*

$$g_x(x^*(\theta), X^*(\theta), \theta) = c_x(x^*(\theta)) \tag{12}$$

*and*

$$\left( g(x^*(\theta), X^*(\theta), \theta) - c(x^*(\theta)) \right) h_y(y^*(\theta), Y^*(\theta), \theta) = k_y(y^*(\theta), Y^*(\theta), \theta). \tag{13}$$

A set of sufficient conditions on the primitives of our model for the unconstrained solution to be optimal does exist, but is omitted in this extended abstract. The idea for these conditions is to impose supermodularity on the integrand to obtain, following Milgrom and Shannon (1994), monotonicity in the parameter. Once an optimal fulfilled-expectations attribute schedule $(x^*, y^*)$ has been determined, it is straightforward to compute the optimal nonlinear pricing schedule as the last piece of the firm's optimal multiattribute menu of network goods.

Proposition 6 (Optimal Nonlinear Multiattribute Pricing) *The optimal nonlinear pricing schedule* $t^*$ *is given by*

$$t^*(\theta) = t_0^* + \int_{\underline{\theta}}^{\theta} \left( \langle g_x^*(\vartheta) - c_x^*(\vartheta), x^{*\prime}(\vartheta) \rangle + \frac{g^*(\vartheta) - c^*(\vartheta)}{h^*(\vartheta)} \langle h_y^*(\vartheta), y^{*\prime}(\vartheta) \rangle \right) d\vartheta, \tag{14}$$

*where*

$$t_0^* = \frac{\int_\Theta \left[ g^*(\theta)h^*(\theta) - \int_{\underline{\theta}}^{\theta} \left( \langle g_x^*(\vartheta) - c_x^*(\vartheta), x^{*\prime}(\vartheta) \rangle + \frac{g^*(\vartheta)-c^*(\vartheta)}{h^*(\vartheta)} \langle h_y^*(\vartheta), y^{*\prime}(\vartheta) \rangle \right) d\vartheta \right] dF(\theta) - u_0}{\int_\Theta h^*(\theta) dF(\theta)},$$

*with the abbreviation* $g^*(\theta) = g(x^*(\theta), X^*(\theta), \theta)$ *and analogous conventions for* $g_x^*, c^*, c_x^*, h^*, h_y^*$.

In the paper we provide a number of concrete examples and applications of our results.

## 3 Discussion

It is remarkable that in contrast to the standard single-dimensional screening problem, the optimal attribute menu $(x^*, y^*)$ does not have to be monotonic in $\theta$. Thus, inefficient bunching (i.e., providing the same product to consumers of different types, effectively failing to separate them) can often be avoided. The likelihood of bunching decreases, thus increasing the firm's rent extraction capability as the firm adds more attributes to its network goods.

We note that in the unconstrained solution to the multiattribute screening problem the product design is independent of consumption probability. Thus direct fulfilled-expectation externalities can be determined independently of the indirect externalities (provided, of course, that $X$ does not depend on $y$). Applications of our results abound, since most goods can be interpreted as network goods. Our approach may also provide a first approximation for the design of a discrete $n$-product line to screen a continuous type distribution, a problem that is riddled with nonconvexities. Further work may look at multiattribute nonlinear pricing of network goods in an oligopolistic environment.

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